EDUCATIONAL MONOGRAPH

prepared from

Design of Time-Weighted Minimum Energy
Discrete-Data Control Systems
(NASA Acc. No. N65-16268)

by

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FOREWORD

This monograph was produced at Virginia Polytechnic Institute in a pilot program administered by Oklahoma State University under contract to the NASA Office of Technology Utilization. The program was organized to determine the feasibility of presenting the results of recent research in NASA Laboratories, and under NASA contract, in an educational format suitable as supplementary material in classwork at engineering colleges. The monograph may result from editing single technical reports or synthesizing several technical reports resulting from NASA's research efforts.

Following the preparation of the monographs, the program includes their evaluation as educational material in a number of universities throughout the country. The results of these individual evaluations in the classroom situation will be used to help determine if this procedure is a satisfactory way of speeding research results into engineering education.

ABSTRACT

This monograph, based on NASA Report N65-16268 [1], discusses a technique for the design of minimum energy discrete-data control systems. The "derived" matrix is used to determine the control sequence that will take the state of plant from some initial state to a desired final state in N sampling periods. The cost function stresses the controlling action of any part of the input sequence and relegates the remainder of the sequence to a supporting role.

INSTRUCTORS' GUIDE FOR MONOGRAPHS

1. Educational level of the monograph--Graduate course in optimal control of discrete data systems.
2. Prerequisite course material--The students should understand the use of the "canonical" vectors and sampled-data control systems.
3. Estimated lecture time required--One hour.
4. Technical significance--The material presents the use of the "derived" matrix to determine the minimum-energy control sequence.
5. New or unusual concepts--The "derived" matrix.
6. How monographs can best be used--It is suggested that:
   (a) A lecture of approximately one hour be given over the monograph material.
   (b) The class be assigned the home problem.
7. Other reports reviewed by the editor in preparing this Monograph--Revington, A. M. and Hung, J. C.: Design of Minimum Energy Discrete-Data Control Systems, Department of Electrical Engineering, The University of Tennessee, NASA N65-16439.
8. Note to instructor: All uncolored pages of the instructors monograph are in the copies intended for student use.

DESIGN OF TIME-WEIGHTED MINIMUM ENERGY DISCRETE-DATA CONTROL SYSTEMS

Consider an n-th order, linear, time-invariant, sampled-data system. The state transition equation for the system is

$$\underline{x}\,[(k+1)\,T] = \Phi(T)\underline{x}(kT) + \underline{h}(T)m\,[(k+1)T], \tag{1}$$

where $\underline{x}(kT)$ is the state vector at time kT and T is the sampling period. Also, $\Phi(T)$ is the nxn state transition matrix, $\underline{h}(T)$ is the n-dimensional forcing vector and m(kT) is the control during the interval (k-1)T, kT.

The object of the optimization is to determine the input sequence m(kT), k=1, 2, ..., N that will take the system from any given initial state $\underline{x}_0$ to the origin of the state space X in N sampling periods and at the same time minimize the cost function

$$E = \sum_{k=1}^{N} d(k)\,[m(kT)]^2 , \tag{2}$$

where the d(k) are positive scalars. When d(k)=1, for all k, this problem is the minimum energy problem.

Mathematical Development

The canonical vectors [2] for the system of equation (1) are

$$\underline{r}_i = \Phi^{-1}(T)\,\underline{h}(T) = \Phi(-iT)\,\underline{h}(T), \quad i = 1,2,\ldots,N, \tag{3}$$

where N > n. If the system is completely controllable, the first n $\underline{r}_i$'s can be used to form a basis for the n-dimensional state space X and any state $\underline{x}$ in X can be expressed in the form

$$\underline{x} = \sum_{k=1}^{N} a_i\underline{r}_i , \tag{4}$$

where the a_i are real constants.

Kalman and Bertram have shown [4,5] that the state $\underline{x}$ can be taken to the origin of the state space by applying the control sequence

$$m(kT) = -a_k \quad k = 1,2,\ldots,N. \tag{5}$$

If N=n, the solution of the n simultaneous equations of equation (4) is the unique solution to the linear time-optimal regulator problem.

If N > n, only n of the N $\underline{r}_i$ vector are linearly independent and there are an infinite number of input sequences which will take $\underline{x}$ to the origin in N sampling periods. The object of this minimization is to determine which of the infinite control sequences also minimizes equation (2). If N > n, equation (4) can be written in the form

$$\underline{x} = \sum_{k=1}^{n} a_K \underline{r}_K + \sum_{k=1}^{N-n} b_k \underline{r}_{n+k} \quad , \tag{6}$$

or

$$\underline{x} = R\underline{a} + Q\underline{b} \; , \tag{7}$$

where R is an nxn matrix with the n canonical vectors $\underline{r}_k$, k=1,...,n, as columns. Q is an nx(N-n) matrix with the canonical vectors $\underline{r}_{n+1}$, $\underline{r}_{n+2}$, ..., $\underline{r}_N$ as columns. Except for a change in sign, the components of $\underline{a}$ and $\underline{b}$ represent the input sequence to be applied to the plant. Pre-multiplying equation (7) by R^{-1} gives

$$R^{-1}\underline{x} = \underline{a} + R^{-1}Q\underline{b}. \tag{8}$$

If the definitions

$$\underline{c} = R^{-1}\underline{x} \tag{9}$$

and

$$H = R^{-1}Q \tag{10}$$

are substituted into equation (8), $\underline{a}$ can be expressed in the form

$$\underline{a} = \underline{c} - H\underline{b}. \tag{11}$$

The nx(N-n) matrix H is the "derived matrix", which is "derived" from the last N-n canonical vectors. Equation (9) transforms the vector $\underline{x}$ of state space X into the vector $\underline{c}$ of "canonical vectors space" C. The coordinates of state space C are the first n canonical vectors r_k, k=1,...,n.

The energy consumption can be written in the compact form

$$E = \underline{a}^T D\underline{a} + \underline{b}^T F\underline{b}, \tag{12}$$

where D is the diagonal matrix with elements $d_{kk} = d(k)$, k=1,2,...,n, and F is the diagonal matrix with elements $f_{kk} = d(k+n)$, k=1,2,...,N-n. Substituting equation (11) into equation (12) gives

$$E = (\underline{c} - H\underline{b})^T D(\underline{c} + H\underline{b}) + \underline{b}^T F\underline{b}, \tag{13}$$

or

$$E = \underline{c}^T D\underline{c} - 2\, c^T DH\underline{b} + \underline{b}^T [H^T DH + F]\underline{b}, \tag{14}$$

which expresses the energy E as a function of N-n independent variables from $\underline{b}$. In order to minimize E, set $\frac{\partial E}{\partial \underline{b}} = 0$. The condition for a minimum is

$$[H^T DH + F]\underline{b} = H^T D\underline{c}\ , \tag{15}$$

and $\frac{\partial E}{\partial \underline{b}^2}$ positive. Equation (15) can be written in terms of $\underline{a}$ and $\underline{b}$ of equation (15),

$$H^T DH\underline{b} + F\underline{b} = H^T D\underline{a} + H^T DH\underline{b}. \tag{16}$$

From equation (16), the conditions for a minimum reduces to,

$$F\underline{b} = H^T D\underline{a} \tag{17}$$

and this can be substituted into equation (12) to give the minimal value of energy

$$E_m = \underline{a}^T D\underline{a} + B^T H^T D\underline{a} = [\underline{a}^T + \underline{b}^T H^T]\, D\underline{a}. \tag{18}$$

Equation (17) can also be substituted into equation (11) to give

$$[I + HF^{-1}H^T D]\underline{a} = \underline{c}\ , \tag{19}$$

where I is the identity matrix. Now define

$$B = I + HF^{-1}H^T D \tag{20}$$

and equation (18) can be written in the form

$$\underline{a} = B^{-1}\underline{c}. \tag{21}$$

The existence of B^{-1} can be shown [2]. Equations (18) and (19) can be combined to reduce the expression for minimal energy to

$$E_m = \underline{c}^T D\underline{a}. \tag{22}$$

Thus, the condition for minimal energy as well as its value can be expressed in a compact form. The essential steps of the process are outlined in the following summary.

Summary

The system state vector x(kT) expressed in terms of the system canonical vector $\underline{r}_i$, i = 1,2,...,N, and the vectors $\underline{a}$ and $\underline{b}$ which represent the input sequence to be applied to the plant. The "derived" matrix H is used to relate $\underline{a}$ to $\underline{b}$ to the vector $\underline{c}$ of "canonical vector space" C,

$$\underline{c} = \underline{a} + H\underline{b} \tag{23}$$

The energy required to take $\underline{c}$ to the origin is

$$E = \underline{a}^T D \underline{a} + \underline{b}^T F \underline{b}, \tag{24}$$

and is minimized when

$$\underline{b} = F^{-1} H^T D \underline{a}. \tag{25}$$

The minimal value of energy is

$$E = -\underline{c}^T D \underline{a} \tag{26}$$

where

$$\underline{a} = B^{-1} \underline{c}. \tag{27}$$

The desired control is

$$[m(1), m(2), \ldots, m(n)] = [-a_1, -a_2, \ldots, -a_n] = -\underline{a}^T \tag{28}$$

$$[m(n+1), m(n+2), \ldots, m(N)] = [-b_1, -b_2, \ldots, -b_{N-n}] = -\underline{b}^T \tag{29}$$

Design Procedure

1) Determine the transition matrix Φ and the forcing vector $\underline{n}$.
2) Determine the canonical vectors $\underline{r}$.
3) Determine the matrices R and Q.
4) Determine the "derived" matrix H.
5) Determine the canonical state vector $\underline{c}$.
6) Determine the matrix B using desired values for F and D.
7) Determine $\underline{a}$ from $\underline{c}$ and B.
8) Determine $\underline{b}$ from $\underline{a}$.
9) Determine the control m(kT) from the components of $\underline{a}$ and $\underline{b}$.
10) Check energy consumed.

Conclusion

A procedure has been presented for the time weighted minimum energy discrete-data control of an n-th order plant. The procedure utilizes the "derived" matrix to simplify the calculations. The procedure is applicable to systems of any order and type.

Home Problem

Consider the second-order system

$$\frac{Y(s)}{M(s)} = \frac{1}{s(s+1)}, \tag{30}$$

where the input M(s) is the output of a zero order hold with T = 1 second and the system output is Y(s). Let the state variable be $x_1(t) = y(t)$ and $x_2(t) = y(t)$. The initial conditions are $x_1(0) = 1.0$ and $x_2 = (0) = 0.0$.

(a) Determine the state transition equation.

(b) Determine the control sequence that drives the system to the origin in four sampling periods and minimizes energy. Let D = F = I (no time weighting).

(c) Determine the control sequence that drives the system to the origin in four sampling periods and minimizes the time-weighted energy. Let

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } F = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}.$$

References

1. Revington, A. M. and J. C. Hung, "Design of Time Weighted Minimum Energy Discrete-Data Control Systems", Department of Electrical Engineering, University of Tennessee, Research Report, NASA, N65-16268.
2. Revington, A. M., and J. C. Hung, "Design of Minimum Energy Discrete-Data Control Systems", Department of Electrical Engineering, University of Tennessee, Research Report, NASA, N65-16439.
3. Revington, A. M., and J. C. Hung, "On Minimum Fuel and Energy Control of

Sampled-Data Control Systems", Department of Electrical Engineering, University of Tennessee, Research Report, NASA, N66-27320.

4. Kalman, R. E., and J. E. Bertram, "General Synthesis Procedure for Computer Control of Single Loop and Multiloop Linear Systems", Trans. AIEE, Vol. 77, Part II, pp. 602-609, 1959.
5. Kalman, R. E., "Optimum Nonlinear Control of Saturating Systems by Intermittent Action", IRE WESCON RECORD, Vol. 1, Part 5, pp. 130-135, 1957.

Solution of Home Problem

(a) Determine the state transition equation. The vector differential equation is

$$\underline{\dot{x}}(t) = A\,\underline{x}(t) + \underline{g}(t)\,m(t), \tag{31}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \text{ and } \underline{g}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The state transition equation is

$$\underline{x}(t) = \Phi(t)\underline{x}(0) + \underline{h}(t)m(t), \tag{32}$$

where

$$\Phi(t) = \begin{bmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{bmatrix} \text{ and } \underline{h}(t) = \begin{bmatrix} e^{-t} + t-1 \\ 1 - e^{-t} \end{bmatrix}.$$

(b)(c) Determination of minimal energy control sequence for N = 4. The canonical vectors are

$$\underline{r}_k = \begin{bmatrix} -(e^k - e^{k-1} - 1) \\ e^k - e^{k-1} \end{bmatrix} \quad \text{for } k = 1,2, \ldots, 4,$$

and

$$R = \begin{bmatrix} -0.7182 & -3.6706 \\ 1.7182 & 4.6706 \end{bmatrix}$$

The matrix Q is

$$Q = \begin{bmatrix} -11.6961 & -33.5118 \\ 12.6961 & 34.5118 \end{bmatrix}$$

and the derived matrix H is

$$H = \begin{bmatrix} -2.7183 & 10.107 \\ 3.7183 & 11.107 \end{bmatrix}$$

The vector $\underline{c}$ is

$$\underline{c} = \begin{bmatrix} 1.582 \\ -0.582 \end{bmatrix},$$

and the elements of the matrix B are

$$b_{11} = 1 + d_{11}\left[\frac{(2.7183)^2}{f_{11}} + \frac{(10.107)^2}{f_{22}}\right]$$

$$b_{12} = d_{22}\left[\frac{-(2.7133)(3.7183)}{f_{11}} - \frac{(10.107)(11.107)}{f_{22}}\right]$$

$$b_{21} = d_{11}\left[\frac{-(2.7183)(3.7183)}{f_{11}} - \frac{(10.107)(11.107)}{f_{22}}\right]$$

$$b_{22} = 1 + d_{22}\left[\frac{(3,6183)^2}{f_{11}} + \frac{(11.107)^2}{f_{22}}\right]$$

For part (b), (No time weighting) D = F = 1 and $\underline{a} = B^{-1}\underline{c}$, which is

$$a = \begin{bmatrix} 0.4875 \\ 0.4275 \end{bmatrix}.$$

For the vector $\underline{b}$, use $\underline{b} = H^T\underline{a}$ which is

$$\underline{b} = \begin{bmatrix} 0.2643 \\ -0.1794 \end{bmatrix}.$$

The control sequence is therefore

m(1) = -0.4875
m(2) = -0.4275
m(3) = -0.2643
m(4) = 0.1794

and the minimum input energy is

$$E = 0.05226.$$

For part C, (time weighting),

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } F = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}.$$

The vectors $\underline{a}$ and $\underline{b}$ are

$$\underline{a} = \begin{bmatrix} 0.7078 \\ 0.3014 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 0.1056 \\ -0.1149 \end{bmatrix}$$

The control sequence for the time weighted case is

$m(1) = -0.7078$

$m(2) = -0.3014$

$m(3) = -0.1056$

$m(4) = 0.1149$

The difference in the required input energy should be pointed out. If time-optimal control were used, only two sampling periods would have been required but the input energy would be much greater.

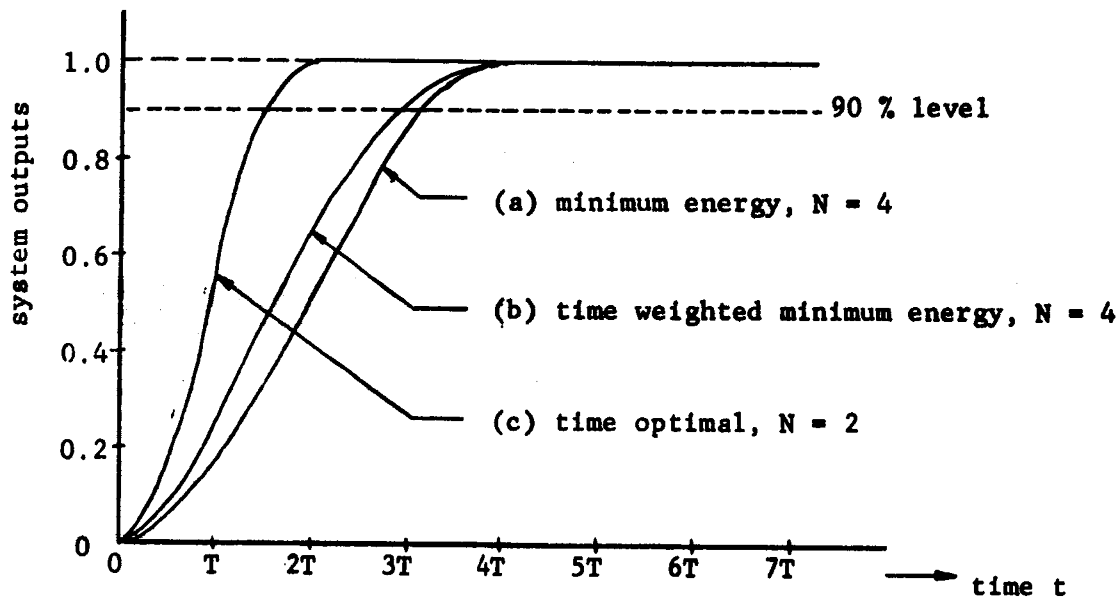

Fig. 1. System output responses

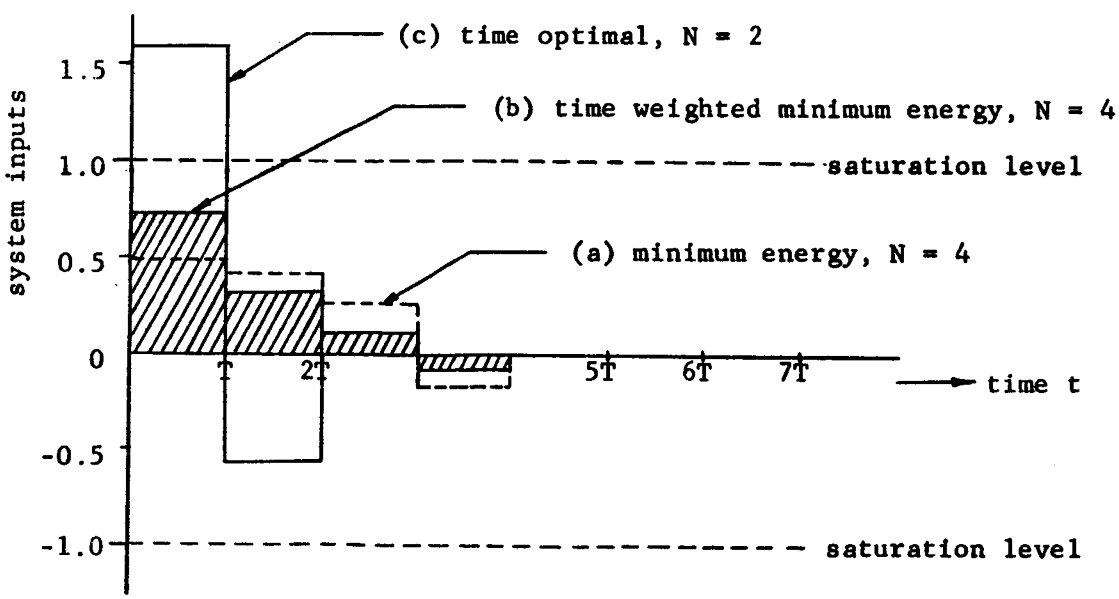

Fig. 2. System inputs